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Constructing quantum games from a system of Bell's inequalities

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Abstract

We report constructing quantum games directly from a system of Bell's inequalities using Arthur Fine's analysis published in early 1980s. This analysis showed that such a system of inequalities forms a set of both necessary and sufficient conditions required to find a joint distribution function compatible with a given set of joint probabilities, in terms of which the system of Bell's inequalities is usually expressed. Using the setting of a quantum correlation experiment for playing a quantum game, and considering the examples of Prisoners' Dilemma and Matching Pennies, we argue that this approach towards constructing quantum games addresses some of their well known criticisms.

Keywords: quantum games, Prisoner's Dilemma, Matching Pennies, Nash equilibrium, quantum correlation experiments, joint probability, quantum probability

1 Introduction

Recent years have witnessed a growing interest in the area of quantum games [1–47]. It seems that there is now an agreement that construction of a quantum game can be achieved along several possible routes [3, 5, 13, 18, 27, 35, 40, 47]. The unifying idea underpinning these routes appears to consist of establishing a link between the classical feature of a physical system, shared by participating players and that facilitates the physical implementation of the game, and the classical game, so that a classical game results because of those features. It turns out that this link takes the form of constraints placed on the (statistical) properties of the shared physical system. A quantum game is then obtained by replacing the classical features of the shared physical system with the quantum ones, while retaining the mentioned link. Subsequently, one investigates the impact the quantum feature(s) may have on the solution/outcome of the game.

As the mentioned link between the classical features of shared physical system and the resulting classical game can be established in several possible ways, there can exist many routes in obtaining a quantum game. In the context of non-cooperative games this results in an interesting situation that one route to quantize a game can provide a new quantum mechanical Nash equilibrium (NE[1]) that is different from the one which another quantization route provides. In Bleiler's words [40] comparing one route to another is similar to comparing apples and oranges.

For instance, in Eisert et al.'s scheme [5] for quantizing a game, the classical game corresponds to an initial product quantum state, which the players share for its local unitary manipulation and its later measurement. In Marinatto and Weber's (MW) quantization scheme [13] the classical game corresponds to the product initial state $|00\rangle$, as a pure state is forwarded to player for its local manipulation by the identity $\hat{I}$ and Pauli $\hat{\sigma}_x$ operators before the final state is measured.

The link that Eisert et al.'s quantization scheme establishes between the classical game and a product initial state (representing the 'classicality' of shared physical system) does not seem entirely convincing. It is because a product initial state leads to a classical game but the classical game can also result when players locally maneuver a maximally entangled state with special

[1]In the rest of this paper we use NE to mean Nash Equilibrium or Nash Equilibria. The correct meaning is to be judged from the context.

unitary actions. Likewise, the relationship which the MW's scheme establishes between the classical game and the product initial state $|00\rangle$ does not appear convincing as a quantum game can correspond to a pure initial state that does not violate Bell's inequality.

It is true that in these quantization schemes a classical game is embedded in the quantum game, but the respective embeddings do not ensure that a quantum game results only when relevant Bell's inequalities are violated. Along with these observations, Benjamin and Hayden [8] pointed out that in the Eisert et al.'s quantization scheme players' unitary actions are arbitrarily restricted and are not even closed under composition. Also, referring to the same quantization formalism, Flitney and Hollenberg [37] pointed out that the new NE and the classical-quantum transitions that occur are simply an artifact of the particular strategy space chosen. Also, Enk and Pike's [10] remarked that as the scheme involves players having access to strategy sets that are not available to them in the classical game, it makes sense to equate the quantum game to an extended classical game constructed by adding extra pure strategies in the game matrix.

In an effort to reply to these observations we proposed [33, 35, 43, 44] the setting of a quantum correlation experiment [48–53] for playing a quantum game. The quantum game is played between two remotely located agents/observers Alice and Bob who can perform measurements on parts of a particle that has disintegrated into two. Alice and Bob each are given two directions in which measurements are performed. In a run, each agent performs a measurement along one of the two directions, whose outcome is a dichotomic variable. A players' strategy is the probability distribution of choosing between the two available directions.

A quantum correlation experiment seems to provide a natural setting for a two-player two-strategy (2×2) quantum game. It is because a player has to decide a pure strategy in each run and the probability distribution over pure strategies defines a player's (mixed) strategy that is definable over many runs. Referring to the quantum correlation experiment, we can identify the agents Alice and Bob as players and assign the two available directions to correspond to a player's pure strategies. Players' payoffs are then expressed in terms of the joint probabilities relevant to the shared physical systems, their moves or strategies, and the entries in the matrix that defines the game.

In the present paper we present a new approach for construction of quantum games that, once again, uses the setting of quantum correlation experiments. We propose that a quantum game corresponding to a classical game should satisfy the following two requirements: a) so as to avoid Enk and Pike type argumentation [10] the strategy sets available to the players are identical in both the quantum and the corresponding classical game, b) the quantization procedure should establish a convincing relation between the classicality of the shared physical system, as it is expressed by a relevant system of Bell's inequalities [61], and the classical game. We show that when the classicality of a shared physical system is defined [61] in terms of a system of Bell's inequalities it allows us to establish such a relationship. We refer, in this connection, to the results reported by Fine [63] in early 1980s and build up our arguments on them. These results are known to be significant with reference to joint distributions, quantum correlations, and a system of Bell's inequalities, all of which are relevant to the setting of quantum correlation experiments that we use to play quantum games.

2 Two-player two-strategy games

Consider a two-player two-strategy game

$$\begin{array}{ccc} & & \text{Bob} \\ & & \begin{array}{cc} S'_1 & S'_2 \end{array} \\ \text{Alice} & \begin{array}{c} S_1 \\ S_2 \end{array} & \left[\begin{array}{cc} (a_1, b_1) & (a_2, b_2) \\ (a_3, b_3) & (a_4, b_4) \end{array} \right], \end{array} \tag{1}$$

in which $S_{1,2}$ and $S'_{1,2}$ are Alice's and Bob's pure strategies, respectively, whereas a_i are Alice's and b_j are Bob's payoffs. For instance, we use $\Pi_A(S_2, S'_1)$ to denote Alice's payoff when she plays

S_2 while Bob plays S_1', which is a_3 from the Table (1). In a mixed-strategy game, we denote by x the probability with which Alice chooses her pure strategy S_1. She then chooses S_2 with the probability $(1-x)$. Similarly, we denote by y the probability with which Bob chooses S_1'. He then chooses S_2' with the probability $(1-y)$. In this case we write Alice's payoff by $\Pi_A(x,y)$ and Bob's payoff by $\Pi_B(x,y)$ i.e. the first entry in bracket is for Alice and the second for Bob. For a symmetric game we have $a_1 = b_1$, $a_4 = b_4$, $a_2 = b_3$, and $a_3 = b_2$, for which one obtains $\Pi_A(x,y) = \Pi_B(y,x)$. The inequalities

$$\Pi_A(x^\star, y^\star) - \Pi_A(x, y^\star) \geqslant 0, \ \Pi_B(x^\star, y^\star) - \Pi_B(x^\star, y) \geqslant 0, \tag{2}$$

describe that the strategy pair $(x^\star, y^\star)$ is a NE.

3 Quantum games using correlation experiments

In the setting of quantum correlation experiments [35], which we use to play a quantum version of the game (1), players Alice and Bob are located in space-time regions R_1 and R_2, respectively. Ideally these regions are spacelike separated. Alice can perform measurements on two bivalent observables (with values ± 1) A_1 and A_2 in region R_1. Similarly, player Bob can perform measurements on two bivalent observables (with values ± 1) B_1 and B_2 in region R_2.

Referring to the matrix (1) we make the associations $A_1 \sim S_1$, $A_2 \sim S_2$ and $B_1 \sim S_1'$, $B_2 \sim S_2'$ and take

$$\triangle_1 = (a_3 - a_1), \ \triangle_2 = (a_4 - a_2), \ \triangle_3 = (\triangle_2 - \triangle_1). \tag{3}$$

We then consider the joint probabilities P_{A_i,B_j} (for $i = 1,2$ and $j = 1,2$) and denote $P(A_1 B_2)$, for example, the probability that both the observable A_1 and B_2 take the value $+1$. Similarly, we denote $P(A_1 \bar{B}_2)$ for the joint probability when the observable A_1 takes the value $+1$ and the observable B_2 takes the value -1.

For the matrix game (1) played in the setting of quantum correlation experiments the payoff relations are expressed [35] as

$$\Pi_{A,B}(x,y) = \begin{bmatrix} x \\ 1-x \end{bmatrix}^T \begin{bmatrix} \Pi_{A,B}(S_1, S_1') & \Pi_{A,B}(S_1, S_2') \\ \Pi_{A,B}(S_2, S_1') & \Pi_{A,B}(S_2, S_2') \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix}, \tag{4}$$

where T is transpose and subscripts A and B refer to Alice and Bob, respectively. In (4) we define

$$\begin{aligned}
\Pi_{A,B}(S_1, S_1') &= (a,b)_1 P(A_1 B_1) + (a,b)_2 P(A_1 \bar{B}_1) + (a,b)_3 P(\bar{A}_1 B_1) + (a,b)_4 P(\bar{A}_1 \bar{B}_1), \\
\Pi_{A,B}(S_1, S_2') &= (a,b)_1 P(A_1 B_2) + (a,b)_2 P(A_1 \bar{B}_2) + (a,b)_3 P(\bar{A}_1 B_2) + (a,b)_4 P(\bar{A}_1 \bar{B}_2), \\
\Pi_{A,B}(S_2, S_1') &= (a,b)_1 P(A_2 B_1) + (a,b)_2 P(A_2 \bar{B}_1) + (a,b)_3 P(\bar{A}_2 B_1) + (a,b)_4 P(\bar{A}_2 \bar{B}_1), \\
\Pi_{A,B}(S_2, S_2') &= (a,b)_1 P(A_2 B_2) + (a,b)_2 P(A_2 \bar{B}_2) + (a,b)_3 P(\bar{A}_2 B_2) + (a,b)_4 P(\bar{A}_2 \bar{B}_2).
\end{aligned} \tag{5}$$

where $(a,b)_2$, for example, is shortened notation for a_2, b_2—the entries in the matrix (1).

Although players' payoffs (4) depend on the joint probabilities corresponding to the shared physical system, the players' moves, represented by x and y, are independent of them. Players' moves in the quantum game are classical in being a linear combination (with real & normalized coefficients) of the two choices available to each player. However, in contrast to the situation in a classical game (in which the chosen strategies directly determine the payoff entries in the payoff matrix), our setting demands that players' payoffs not only depend on their moves but also that these depend on what kind of physical system players share in order to play the game. We achieve this by making the payoff relations to depend also on the joint probabilities relevant to the shared physical system and then ask whether a joint probability distribution exists. As for a quantum

mechanical shared physical system the joint probabilities can go beyond the constraints permitted to classical joint probabilities, allowing us to obtain our quantum game.

As the joint probabilities are normalized we have

$$\begin{aligned}
&P(A_1B_1)+P(A_1\bar{B}_1)+P(\bar{A}_1B_1)+P(\bar{A}_1\bar{B}_1)=1,\\
&P(A_1B_2)+P(A_1\bar{B}_2)+P(\bar{A}_1B_2)+P(\bar{A}_1\bar{B}_2)=1,\\
&P(A_2B_1)+P(A_2\bar{B}_1)+P(\bar{A}_2B_1)+P(\bar{A}_2\bar{B}_1)=1,\\
&P(A_2B_2)+P(A_2\bar{B}_2)+P(\bar{A}_2B_2)+P(\bar{A}_2\bar{B}_2)=1,
\end{aligned} \tag{6}$$

and thus each one of the relations (5) represents a classical mixed strategy payoff. The causal communication constraint [53] for the joint probabilities P_{A_i,B_j} (for $i=1,2$ and $j=1,2$) now states that

$$\begin{aligned}
P(A_1B_1)+P(A_1\bar{B}_1) &= P(A_1B_2)+P(A_1\bar{B}_2),\ P(A_1B_1)+P(\bar{A}_1B_1)=P(A_2B_1)+P(\bar{A}_2B_1),\\
P(A_2B_1)+P(A_2\bar{B}_1) &= P(A_2B_2)+P(A_2\bar{B}_2),\ P(A_1B_2)+P(\bar{A}_1B_2)=P(A_2B_2)+P(\bar{A}_2B_2),\\
P(\bar{A}_1B_1)+P(\bar{A}_1\bar{B}_1) &= P(\bar{A}_1B_2)+P(\bar{A}_1\bar{B}_2),\ P(\bar{A}_2B_1)+P(\bar{A}_2\bar{B}_1)=P(\bar{A}_2B_2)+P(\bar{A}_2\bar{B}_2),\\
P(A_1\bar{B}_1)+P(\bar{A}_1\bar{B}_1) &= P(A_2\bar{B}_1)+P(\bar{A}_2\bar{B}_1),\ P(A_1\bar{B}_2)+P(\bar{A}_1\bar{B}_2)=P(A_2\bar{B}_2)+P(\bar{A}_2\bar{B}_2).
\end{aligned} \tag{7}$$

Using Eqs. (6,7) it can be shown [53] that 8 out of 16 joint probabilities P_{A_i,B_j} (for $i=1,2$ and $j=1,2$) can be eliminated.

With the payoff relations (4) the Nash inequalities for an arbitrary pair of strategies $(x^\star,y^\star)$ are written as

$$\begin{aligned}
\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star) \;=\; & (x^\star-x)\{y^\star\{a_1[P(A_1B_1)-P(A_1B_2)-P(A_2B_1)+P(A_2B_2)]+\\
& a_2[P(A_1\bar{B}_1)-P(A_1\bar{B}_2)-P(A_2\bar{B}_1)+P(A_2\bar{B}_2)]+\\
& a_3[P(\bar{A}_1B_1)-P(\bar{A}_1B_2)-P(\bar{A}_2B_1)+P(\bar{A}_2B_2)]+\\
& a_4[P(\bar{A}_1\bar{B}_1)-P(\bar{A}_1\bar{B}_2)-P(\bar{A}_2\bar{B}_1)+P(\bar{A}_2\bar{B}_2)]\}+\\
& \{a_1[P(A_1B_2)-P(A_2B_2)]+a_2[P(A_1\bar{B}_2)-P(A_2\bar{B}_2)]+\\
& a_3[P(\bar{A}_1B_2)-P(\bar{A}_2B_2)]+a_4[P(\bar{A}_1\bar{B}_2)-P(\bar{A}_2\bar{B}_2)]\}\}\geq 0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y) \;=\; & (y^\star-y)\{x^\star\{b_1[P(A_1B_1)-P(A_1B_2)-P(A_2B_1)+P(A_2B_2)]+\\
& b_2[P(A_1\bar{B}_1)-P(A_1\bar{B}_2)-P(A_2\bar{B}_1)+P(A_2\bar{B}_2)]+\\
& b_3[P(\bar{A}_1B_1)-P(\bar{A}_1B_2)-P(\bar{A}_2B_1)+P(\bar{A}_2B_2)]+\\
& b_4[P(\bar{A}_1\bar{B}_1)-P(\bar{A}_1\bar{B}_2)-P(\bar{A}_2\bar{B}_1)+P(\bar{A}_2\bar{B}_2)]\}+\\
& \{b_1[P(A_2B_1)-P(A_2B_2)]+b_2[P(A_2\bar{B}_1)-P(A_2\bar{B}_2)]+\\
& b_3[P(\bar{A}_2B_1)-P(\bar{A}_2B_2)]+b_4[P(\bar{A}_2\bar{B}_1)-P(\bar{A}_2\bar{B}_2)]\}\}\geq 0.
\end{aligned} \tag{9}$$

Notice that with respect to the joint probability distribution P_{A_1,A_2,B_1,B_2}, if it exists, the given joints P_{A_i,B_j} (for $i=1,2$ and $j=1,2$) can be expressed as their marginals. For instance

$$P(A_2\bar{B}_1)=P(A_1A_2\bar{B}_1B_2)+P(A_1A_2\bar{B}_1\bar{B}_2)+P(\bar{A}_1A_2\bar{B}_1B_2)+P(\bar{A}_1A_2\bar{B}_1\bar{B}_2). \tag{10}$$

Similar expressions can be written for $P(A_iB_j)$, $P(\bar{A}_iB_j)$, and $P(\bar{A}_i\bar{B}_j)$ for $i=1,2$ and $j=1,2$. In the rest of this paper we refer to P_{A_i,B_j} as joint probabilities and to P_{A_1,A_2,B_1,B_2} as the joint probability distribution.

4 Fine's analysis

At this stage we refer to a result reported in early 1980s by Arthur Fine [63] stating that Bell's inequalities form both necessary and sufficient conditions in order to find a joint probability distribution P_{A_1,A_2,B_1,B_2} whose marginals are the joint probabilities P_{A_i,B_j} (for $i = 1,2$ and $j = 1,2$). For the case when Bell's inequalities hold, Fine describes how to find the probability distribution P_{A_1,A_2,B_1,B_2} from the joints probabilities P_{A_i,B_j}, in terms of which the inequalities are usually expressed.

Fine presents two theorems, the first of which states that if A, B, B' are bivalent observables, each mapping into $\{+1,-1\}$ with given joint distributions $P_{A,B}$, $P_{A,B'}$ and $P_{B,B'}$, then the necessary and sufficient condition for the existence of a joint distribution $P_{A,B,B'}$, compatible with the given joints for the pairs, is the satisfaction of following system of inequalities:

$$\begin{aligned} P(A)+P(B)+P(B') &\leq 1+P(AB)+P(AB')+P(BB'), \\ P(AB)+P(AB') &\leq P(A)+P(BB'), \\ P(AB)+P(BB') &\leq P(B)+P(AB'), \\ P(AB')+P(BB') &\leq P(B')+P(AB), \end{aligned} \tag{11}$$

where $P(\cdot)$ denotes the probability that each enclosed observable takes the value $+1$.

Fine's second theorem [63] states that if A_1, A_2, B_1, B_2 are bivalent observables with joint distributions P_{A_i,B_j} (for $i = 1,2$ and $j = 1,2$), then the necessary and sufficient condition for there to exist a joint distribution P_{A_1,A_2,B_1,B_2} compatible with the given joints is that the following system of Bell's inequalities is satisfied:

$$-1 \leq P(A_iB_j)+P(A_iB_{j'})+P(A_{i'}B_{j'})-P(A_{i'}B_j)-P(A_i)-P(B_{j'}) \leq 0, \tag{12}$$

for $i \neq i' = 1,2$ and $j \neq j' = 1,2$. The second theorem becomes particularly relevant as it relates to the setting of quantum correlation experiments that we use in this paper to play a quantum game.

By these theorems Fine finds the joint probability distribution P_{A_1,A_2,B_1,B_2} by letting $n = 1,2$ and $m \neq k = 1,2$ and by setting

$$\gamma = \min\left\{P(A_nB_m)+P(B_k)-P(A_nB_k), P(B_m), P(B_k)\right\}. \tag{13}$$

Afterwards, by defining $P(B_1B_2) = \gamma$ Fine fills the rest of the distribution by letting

$$\begin{aligned} P(\bar{B}_1B_2) &= P(B_1)-\gamma, \\ P(B_1\bar{B}_2) &= P(B_2)-\gamma, \\ P(\bar{B}_1\bar{B}_2) &= 1-P(B_1)-P(B_2)+\gamma. \end{aligned} \tag{14}$$

Fine then defines two quantities α and β as

$$\alpha = P(A_1B_1B_2) = \gamma P(A_1), \quad \beta = P(A_2B_1B_2) = \gamma P(A_2) \tag{15}$$

to find the distributions P_{A_1,B_1,B_2} and P_{A_2,B_1,B_2} as given below.

$$
\begin{aligned}
P(A_1B_1\bar{B}_2) &= P(A_1B_1)-\alpha, \\
P(A_1\bar{B}_1B_2) &= P(A_1B_2)-\alpha, \\
P(A_1\bar{B}_1\bar{B}_2) &= P(A_1)-P(A_1B_1)-P(A_1B_2)+\alpha, \\
P(\bar{A}_1B_1B_2) &= P(B_1B_2)-\alpha, \\
P(\bar{A}_1B_1\bar{B}_2) &= P(B_1)-P(A_1B_1)-P(B_1B_2)+\alpha, \\
P(\bar{A}_1\bar{B}_1B_2) &= P(B_2)-P(A_1B_2)-P(B_1B_2)+\alpha, \\
P(\bar{A}_1\bar{B}_1\bar{B}_2) &= 1-P(A_1)-P(B_1)-P(B_2)+P(A_1B_1)+P(A_1B_2)+P(B_1B_2)-\alpha,
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
P(A_2B_1\bar{B}_2) &= P(A_2B_1)-\beta, \\
P(A_2\bar{B}_1B_2) &= P(A_2B_2)-\beta, \\
P(A_2\bar{B}_1\bar{B}_2) &= P(A_2)-P(A_2B_1)-P(A_2B_2)+\beta, \\
P(\bar{A}_2B_1B_2) &= P(B_1B_2)-\beta, \\
P(\bar{A}_2B_1\bar{B}_2) &= P(B_1)-P(A_2B_1)-P(B_1B_2)+\beta, \\
P(\bar{A}_2\bar{B}_1B_2) &= P(B_2)-P(A_2B_2)-P(B_1B_2)+\beta, \\
P(\bar{A}_2\bar{B}_1\bar{B}_2) &= 1-P(A_2)-P(B_1)-P(B_2)+P(A_2B_1)+P(A_2B_2)+P(B_1B_2)-\beta,
\end{aligned} \tag{17}
$$

from which the distribution P_{A_1,A_2,B_1,B_2} can easily be found.

In the following, while using Fine's analysis, we analyze the quantum games of Prisoners's Dilemma (PD) and Matching Pennies (MP) played in the setting of quantum correlation experiments. We have selected these games because both games have been analyzed earlier in Refs. [35, 44] using the concept of non-factorizable joint probabilities. This will provide us an opportunity to find how the outcomes of these quantum games compare when the games are studied using the present approach built up on Fine's analysis and the approach building up on the joint probabilities becoming non-factorizable. Using the concept of non-factorizable joint probabilities, the quantum PD game produces no new outcome over the classical one. Using the same procedure for quantum MP game, however, results in new non-classical NE when the players share a entangled state that maximally violates the Clauser-Holt-Shimony-Horne (CHSH) sum of correlations.

A second reason to chose these games is that classically each of these two games have only one NE—a pure one for PD and mixed one for MP. As it is the case with the approach using non-factorizable joint probabilities, the present approach, based on Fine's analysis, also uses constraints on joint probabilities that are associated with a particular NE. The situation of having a unique classical NE presents a more easily tractable case when we are considering constraints on quantum mechanical joint probabilities, relative to the case when multiple NE exist for a game and we have to separately consider constraints on joint probabilities for each of them.

5 Analysis of the quantum Prisoners' Dilemma game

We now refer to the matrix (1) and consider the PD game for which we have $a_3 > a_1 > a_4 > a_2$ and as it is a symmetric game we have

$$
\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^T = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \tag{18}
$$

where T is for transpose. Referring to (3) we then have $\triangle_1, \triangle_2 > 0$ and $\Delta_1 = b_2 - b_1$ and $\Delta_2 = b_4 - b_3$. The strategy pair $(x^\star, y^\star) = (0, 0)$ can be shown to emerge as a NE for this game at which we have $\Pi_A(0,0) = a_4$ and $\Pi_B(0,0) = b_4$.

To consider the quantum version of this game played using the setting of quantum correlation experiments, we consider the strategy pair $(x^\star, y^\star) = (0, 0)$ for which we define

$$\triangle_{x^\star=0} = \Pi_A(0,0) - \Pi_A(x,0), \quad \triangle_{y^\star=0} = \Pi_B(0,0) - \Pi_B(0,y). \tag{19}$$

For this strategy pair, we insert from Eqs. (10) into the inequalities (8,9) to obtain

$$\begin{aligned}
\triangle_{x^\star=0} &= x[\triangle_1 \{P(A_1\bar{A}_2B_1B_2) - P(\bar{A}_1A_2B_1B_2) - P(\bar{A}_1A_2\bar{B}_1B_2) + P(A_1\bar{A}_2\bar{B}_1B_2)\} + \\
&\quad \triangle_2 \{P(A_1\bar{A}_2B_1\bar{B}_2) - P(\bar{A}_1A_2B_1\bar{B}_2) - P(\bar{A}_1A_2\bar{B}_1\bar{B}_2) + P(A_1\bar{A}_2\bar{B}_1\bar{B}_2)\}], \\
\triangle_{y^\star=0} &= y[\triangle_1 \{P(A_1A_2B_1\bar{B}_2) - P(A_1A_2\bar{B}_1B_2) - P(\bar{A}_1A_2\bar{B}_1B_2) + P(\bar{A}_1A_2B_1\bar{B}_2)\} + \\
&\quad \triangle_2 \{P(A_1\bar{A}_2B_1\bar{B}_2) - P(A_1\bar{A}_2\bar{B}_1B_2) - P(\bar{A}_1\bar{A}_2\bar{B}_1B_2) + P(\bar{A}_1\bar{A}_2B_1\bar{B}_2)\}],
\end{aligned} \tag{20}$$

that express $\triangle_{x^\star=0}$ and $\triangle_{y^\star=0}$ in terms of the joint probability distribution P_{A_1,A_2,B_1,B_2}.

We now demand that the strategy pair $(0,0)$ emerges as a NE when players share a classical physical system. For this we impose the following two requirements: When the system of Bell's inequalities (12) holds we have: a) Nash inequalities for the strategy pair $(0,0)$ hold i.e. $\triangle_{x^\star=0} \geq 0$, $\triangle_{y^\star=0} \geq 0$ and b) the payoffs for players Alice and Bob at the strategy pair $(0,0)$ are a_4 and b_4, respectively. These requirements ensure that a faithful realization of the original game exists within the quantum game constructed using the setting of quantum correlation experiments.

A keen reader may ask here why the converse situation is not adopted i.e. to require that if $\triangle_{x^\star=0} \geq 0$ and $\triangle_{y^\star=0} \geq 0$ then the system of Bell's inequalities holds. Unfortunately, this is not a valid requirement as the violation of Bell's inequalities does not necessarily lead to a new non-classical NE, and we well may have $\triangle_{x^\star=0} \geq 0$ and $\triangle_{y^\star=0} \geq 0$ even when the system of Bell's inequalities is violated.

These requirements remind us of Fine's second theorem, while Eqs. (16,17) allow us to find the joint probability distribution P_{A_1,A_2,B_1,B_2}. Using Eqs. (16,17) we, therefore, re-express the quantities $\triangle_{x^\star=0}$ and $\triangle_{y^\star=0}$ in Eqs. (20) in terms of the joint probabilities P_{A_i,B_j} (for $i = 1, 2$ and $j = 1, 2$) to obtain

$$\begin{gathered}
P(A_1\bar{A}_2B_1B_2) + P(A_1\bar{A}_2\bar{B}_1B_2) - P(\bar{A}_1A_2B_1B_2) - \\
P(\bar{A}_1A_2\bar{B}_1B_2) = [P(A_1) - P(A_2)]P(B_2), \\
P(A_1\bar{A}_2B_1\bar{B}_2) + P(A_1\bar{A}_2\bar{B}_1\bar{B}_2) - P(\bar{A}_1A_2B_1\bar{B}_2) - \\
P(\bar{A}_1A_2\bar{B}_1\bar{B}_2) = [P(A_1) - P(A_2)][1 - P(B_2)], \\
P(A_1A_2B_1\bar{B}_2) + P(\bar{A}_1A_2B_1\bar{B}_2) - P(A_1A_2\bar{B}_1B_2) - \\
P(\bar{A}_1A_2\bar{B}_1B_2) = [P(B_1) - P(B_2)]P(A_2), \\
P(A_1\bar{A}_2B_1\bar{B}_2) + P(\bar{A}_1\bar{A}_2B_1\bar{B}_2) - P(\bar{A}_1\bar{A}_2\bar{B}_1B_2) - \\
P(A_1\bar{A}_2\bar{B}_1B_2) = [P(B_1) - P(B_2)][1 - P(A_2)],
\end{gathered} \tag{21}$$

and the Nash inequalities for the strategy pair $(0,0)$ then take a simpler form

$$\begin{aligned}
\triangle_{x^\star=0} = x[P(A_1) - P(A_2)][(\triangle_1/\triangle_2 - 1)P(B_2) + 1]\triangle_2 \geq 0, \\
\triangle_{y^\star=0} = y[P(B_1) - P(B_2)][(\triangle_1/\triangle_2 - 1)P(A_2) + 1]\triangle_2 \geq 0,
\end{aligned} \tag{22}$$

giving the first set of constraints on joint probabilities as

$$P(A_1) \geq P(A_2), \quad P(B_1) \geq P(B_2). \tag{23}$$

Similarly, below we translate the requirement b) into the second set of constraints on joint probabilities. Consider the relations (4) to find players' payoffs at the strategy pair $(0,0)$ as

$$\Pi_{A,B}(0,0) = (a,b)_1 P(A_2B_2) + (a,b)_2 P(A_2\bar{B}_2) + (a,b)_3 P(\bar{A}_2B_2) + (a,b)_4 P(\bar{A}_2\bar{B}_2), \tag{24}$$

which, as is the case for a), we express using Eqs. (10) in terms of the joint probability distribution P_{A_1,A_2,B_1,B_2} as

$$\begin{aligned}\Pi_{A,B}(0,0) = (a,b)_1\{&P(A_1A_2B_1B_2) + P(A_1A_2\bar{B}_1B_2) + P(\bar{A}_1A_2B_1B_2) + P(\bar{A}_1A_2\bar{B}_1B_2)\}+ \\ (a,b)_2\{&P(A_1A_2B_1\bar{B}_2) + P(A_1A_2\bar{B}_1\bar{B}_2) + P(\bar{A}_1A_2B_1\bar{B}_2) + P(\bar{A}_1A_2\bar{B}_1\bar{B}_2)\}+ \\ (a,b)_3\{&P(A_1\bar{A}_2B_1B_2) + P(A_1\bar{A}_2\bar{B}_1B_2) + P(\bar{A}_1\bar{A}_2B_1B_2) + P(\bar{A}_1\bar{A}_2\bar{B}_1B_2)\}+ \\ (a,b)_4\{&P(A_1\bar{A}_2B_1\bar{B}_2) + P(A_1\bar{A}_2\bar{B}_1\bar{B}_2) + P(\bar{A}_1\bar{A}_2B_1\bar{B}_2) + P(\bar{A}_1\bar{A}_2\bar{B}_1\bar{B}_2)\}.\end{aligned} \tag{25}$$

Observing that we assume that Bell's inequalities hold, at this stage, once again, we refer to Fine's second theorem and insert the probability distribution P_{A_1,A_2,B_1,B_2} given by Eqs. (16,17) into Eq. (25) in order to re-express it in terms of joint probabilities P_{A_i,B_j} (for $i = 1,2$ and $j = 1,2$). This interestingly leads to obtaining (24) again.

Note that the requirement b) states that when Bell's inequalities hold, Alice's and Bob's payoffs at the strategy pair $(0,0)$ are a_4 and b_4, respectively. To find what constraints this puts on joint probabilities P_{A_i,B_j}, we re-express (24) as

$$\begin{aligned}\Pi_A(0,0) &= (a_2 - a_4)P(A_2) + (a_3 - a_4)P(B_2) + (a_1 - a_2 - a_3 + a_4)P(A_2B_2) + a_4, \\ \Pi_B(0,0) &= (b_2 - b_4)P(A_2) + (b_3 - b_4)P(B_2) + (b_1 - b_2 - b_3 + b_4)P(A_2B_2) + b_4,\end{aligned} \tag{26}$$

and then set $\Pi_A(0,0) = a_4$ and $\Pi_B(0,0) = b_4$ to obtain

$$P(A_2) = 0 = P(B_2), \tag{27}$$

which defines the second set of constraints on joint probabilities.

The above analysis allows us to look forward to the situation when a joint probability distribution P_{A_1,A_2,B_1,B_2} does not exist and/or cannot be found from P_{A_i,B_j}—a situation that corresponds when Bell's inequalities are violated. We will consider this under the assumption that the constraints on P_{A_i,B_j}, that are obtained above and are given by the inequalities (22), continue to hold true. This assumption guarantees that the classical game, with its particular outcome and the corresponding payoffs, emerges when Bell's inequalities hold and thus the classical game remains embedded within the corresponding quantum game.

Fine's analysis is used in above to find the constraints (23,27) that we insert, in the following step, into Eqs. (8,9), that are relevant to a general strategy pair $(x^\star, y^\star)$, to find if a strategy pair that is different from the classical case of $(x^\star, y^\star) = (0,0)$, emerges as a NE when the system of Bell's inequalities (12) is violated and, therefore, a probability distribution P_{A_1,A_2,B_1,B_2} cannot be found whose marginals are P_{A_i,B_j}. This leads us to obtain

$$\begin{aligned}\Pi_A(x^\star, y^\star) - \Pi_A(x, y^\star) &= (x^\star - x)[y^\star(1 - \triangle_1/\triangle_2)P(B_1) - 1]\triangle_2 P(A_1) \geq 0, \\ \Pi_B(x^\star, y^\star) - \Pi_B(x^\star, y) &= (y^\star - y)[x^\star(1 - \triangle_1/\triangle_2)P(A_1) - 1]\triangle_2 P(B_1) \geq 0.\end{aligned} \tag{28}$$

Now, as $\triangle_1, \triangle_2 > 0$ these inequalities once again generate the outcome $(x^\star, y^\star) = (0,0)$. No new NE, therefore, emerges for PD even when Bell's inequalities are violated and the quantum game generates the same outcome as does the classical game.

6 Analysis of the quantum Matching Pennies game

The Matching Pennies (MP) game involves two players Alice and Bob and each player has a penny that s/he secretly flips to heads $\mathcal{H}$ or tails $\mathcal{T}$. Players are not permitted to communicate and they

disclose their strategies to a referee who organizes the game. If the referee finds that the two pennies match (both heads or both tails), he takes one dollar from Bob and gives it to Alice (+1 for Alice, −1 for Bob) and if the pennies mismatch (one heads and one tails), the referee takes one dollar from Alice and gives it to Bob (−1 for Alice, +1 for Bob). As one player's gain is exactly equal to the other player's loss the game is zero-sum with the payoff matrix

$$\begin{array}{cc} & \text{Bob} \\ & \begin{array}{cc} \mathcal{H} & \mathcal{T} \end{array} \\ \text{Alice} \begin{array}{c} \mathcal{H} \\ \mathcal{T} \end{array} & \left(\begin{array}{cc} (+1,-1) & (-1,+1) \\ (-1,+1) & (+1,-1) \end{array} \right). \end{array} \tag{29}$$

No pure strategy NE [55] exists and a unique mixed strategy NE emerges in which both players select the strategies $\mathcal{H}$ and $\mathcal{T}$ with the probability of 1/2. At the strategy pair $(x^\star, y^\star) = (1/2, 1/2)$ players receive $\Pi_A(1/2, 1/2) = 0 = \Pi_B(1/2, 1/2)$.

As in the quantum game the Eq. (4) give the players' payoff relations, for a NE strategy pair $(x^\star, y^\star)$ we obtain

$$\begin{aligned} \Pi_A(x^\star, y^\star) - \Pi_A(x, y^\star) &= [y^\star \{\Pi_A(S_1, S_1') - \Pi_A(S_2, S_1') - \Pi_A(S_1, S_2') + \Pi_A(S_2, S_2')\} \\ &+ \{\Pi_A(S_1, S_2') - \Pi_A(S_2, S_2')\}](x^\star - x) \geq 0, \\ \Pi_B(x^\star, y^\star) - \Pi_B(x^\star, y) &= [x^\star \{\Pi_B(S_1, S_1') - \Pi_B(S_1, S_2') - \Pi_B(S_2, S_1') + \Pi_B(S_2, S_2')\} \\ &+ \{\Pi_B(S_2, S_1') - \Pi_B(S_2, S_2')\}](y^\star - y) \geq 0, \end{aligned} \tag{30}$$

where Eqs. (5) and the matrix (29) gives

$$\begin{aligned} \Pi_A(S_1, S_1') &= P(A_1B_1) - P(A_1\bar{B}_1) - P(\bar{A}_1B_1) + P(\bar{A}_1\bar{B}_1) = -\Pi_B(S_1, S_1'), \\ \Pi_A(S_1, S_2') &= P(A_1B_2) - P(A_1\bar{B}_2) - P(\bar{A}_1B_2) + P(\bar{A}_1\bar{B}_2) = -\Pi_B(S_1, S_2'), \\ \Pi_A(S_2, S_1') &= P(A_2B_1) - P(A_2\bar{B}_1) - P(\bar{A}_2B_1) + P(\bar{A}_2\bar{B}_1) = -\Pi_B(S_2, S_1'), \\ \Pi_A(S_2, S_2') &= P(A_2B_2) - P(A_2\bar{B}_2) - P(\bar{A}_2B_2) + P(\bar{A}_2\bar{B}_2) = -\Pi_B(S_2, S_2'). \end{aligned} \tag{31}$$

The right sides of these Equations express the fact that, as it is the case with the classical game, the quantum game is also zero-sum game.

As it was the case for the PD game, we define

$$\triangle_{x^\star=1/2} = \Pi_A(1/2, 1/2) - \Pi_A(x, 1/2), \quad \triangle_{y^\star=1/2} = \Pi_B(1/2, 1/2) - \Pi_B(1/2, y), \tag{32}$$

and insert from Eqs. (31) into Eqs. (30) to obtain

$$\begin{aligned} \triangle_{x^\star=1/2} = (1/2)[&P(A_1B_1) - P(A_1\bar{B}_1) - P(\bar{A}_1B_1) + P(\bar{A}_1\bar{B}_1) \\ &-P(A_2B_1) + P(A_2\bar{B}_1) + P(\bar{A}_2B_1) - P(\bar{A}_2\bar{B}_1) \\ &P(A_1B_2) - P(A_1\bar{B}_2) - P(\bar{A}_1B_2) + P(\bar{A}_1\bar{B}_2) \\ &-P(A_2B_2) + P(A_2\bar{B}_2) + P(\bar{A}_2B_2) - P(\bar{A}_2\bar{B}_2)](1/2 - x) \geq 0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} \triangle_{y^\star=1/2} = (1/2)[&-P(A_1B_1) + P(A_1\bar{B}_1) + P(\bar{A}_1B_1) - P(\bar{A}_1\bar{B}_1) \\ &P(A_1B_2) - P(A_1\bar{B}_2) - P(\bar{A}_1B_2) + P(\bar{A}_1\bar{B}_2) \\ &-P(A_2B_1) + P(A_2\bar{B}_1) + P(\bar{A}_2B_1) - P(\bar{A}_2\bar{B}_1) \\ &P(A_2B_2) - P(A_2\bar{B}_2) - P(\bar{A}_2B_2) + P(\bar{A}_2\bar{B}_2)](1/2 - y) \geq 0. \end{aligned} \tag{34}$$

We now insert from Eqs. (10) into Eqs. (33, 34) to obtain

$$\triangle_{x^\star=1/2} = 2[P(A_1\bar{A}_2B_1B_2) - P(A_1\bar{A}_2\bar{B}_1\bar{B}_2) - P(\bar{A}_1A_2B_1B_2) + P(\bar{A}_1A_2\bar{B}_1\bar{B}_2)](1/2 - x), \tag{35}$$

$$\triangle_{y^\star=1/2} = 2[P(A_1A_2\bar{B}_1B_2) - P(A_1A_2B_1\bar{B}_2) - P(\bar{A}_1\bar{A}_2\bar{B}_1B_2) + P(\bar{A}_1\bar{A}_2B_1\bar{B}_2)](1/2 - y), \tag{36}$$

that express $\triangle_{x^\star=1/2}$ and $\triangle_{y^\star=1/2}$ in terms of the joint probability distribution P_{A_1,A_2,B_1,B_2}. As it was the case with the PD game, at this stage we demand that the strategy pair $(1/2, 1/2)$ results as a NE when players share a classical physical system, for which the system (12) of Bell's inequalities hold and the joint probability distribution P_{A_1,A_2,B_1,B_2} can be found from P_{A_i,B_j} using Fine's second theorem. It, therefore, seems natural to impose the following two requirements: When the system of Bell's inequalities (12) hold we have: a) Nash inequalities for the strategy pair $(1/2, 1/2)$ hold i.e. $\triangle_{x^\star=1/2} \geq 0$, $\triangle_{y^\star=1/2} \geq 0$, b) the payoffs for players Alice and Bob at the strategy pair $(1/2, 1/2)$ are zero both.

As has been the case with the PD game, a keen reader may ask here why the converse situation is not adopted: to require that if $\triangle_{x^\star=1/2} \geq 0$ and $\triangle_{y^\star=1/2} \geq 0$ then the system of Bell's inequalities holds. As the violation of the system of Bell's inequalities does not necessarily lead to a new non-classical NE, and we well may have $\triangle_{x^\star=1/2} \geq 0$ and $\triangle_{y^\star=1/2} \geq 0$ even when Bell's inequalities are violated, we consider it not to be a valid requirement.

To address a) we require that if the probability distribution P_{A_1,A_2,B_1,B_2}, obtained from Fine's second theorem, is inserted in Eqs. (35,36) then we have $\triangle_{x^\star=1/2} \geq 0$ and $\triangle_{y^\star=1/2} \geq 0$. Inserting from Eqs. (16,17) to Eqs. (35,36) reduces them to simpler form:

$$\begin{aligned}
\triangle_{x^\star=1/2} &= 2[P(A_2) - P(A_1)][1 - P(B_1) - P(B_2)](1/2 - x) \geq 0, \\
\triangle_{y^\star=1/2} &= 2[P(B_1) - P(B_2)][1 - P(A_1) - P(A_2)](1/2 - y) \geq 0,
\end{aligned} \tag{37}$$

which defines the first set of constraints on joint probabilities as

$$[P(A_2) - P(A_1)][1 - P(B_1) - P(B_2)] = 0, \tag{38}$$

$$[P(B_1) - P(B_2)][1 - P(A_1) - P(A_2)] = 0. \tag{39}$$

Now we translate the requirement b) into the second set of constraints on joint probabilities. Consider the relations (4) to find players' payoffs at the strategy pair $(1/2, 1/2)$ as

$$\Pi_{A,B}(1/2, 1/2) = (1/4)\{\Pi_{A,B}(S_1, S_1') + \Pi_{A,B}(S_1, S_2') + \Pi_{A,B}(S_2, S_1') + \Pi_{A,B}(S_2, S_2')\}, \tag{40}$$

where

$$\begin{aligned}
\Pi_A(S_1, S_1') &= P(A_1B_1) - P(A_1\bar{B}_1) - P(\bar{A}_1B_1) + P(\bar{A}_1\bar{B}_1) = -\Pi_B(S_1, S_1'), \\
\Pi_A(S_1, S_2') &= P(A_1B_2) - P(A_1\bar{B}_2) - P(\bar{A}_1B_2) + P(\bar{A}_1\bar{B}_2) = -\Pi_B(S_1, S_2'), \\
\Pi_A(S_2, S_1') &= P(A_2B_1) - P(A_2\bar{B}_1) - P(\bar{A}_2B_1) + P(\bar{A}_2\bar{B}_1) = -\Pi_B(S_2, S_1'), \\
\Pi_A(S_2, S_2') &= P(A_2B_2) - P(A_2\bar{B}_2) - P(\bar{A}_2B_2) + P(\bar{A}_2\bar{B}_2) = -\Pi_B(S_2, S_2').
\end{aligned} \tag{41}$$

Inserting (41) in (40) gives

$$\begin{aligned}
\Pi_A(1/2, 1/2) = (1/4)[\{&P(A_1B_1) + P(A_1B_2) + P(A_2B_1) + P(A_2B_2) + P(\bar{A}_1\bar{B}_1) + \\
&P(\bar{A}_1\bar{B}_2) + P(\bar{A}_2\bar{B}_1) + P(\bar{A}_2\bar{B}_2)\} - \{P(A_1\bar{B}_1) + P(A_1\bar{B}_2) + P(A_2\bar{B}_1) + \\
&P(A_2\bar{B}_2) + P(\bar{A}_1B_1) + P(\bar{A}_1B_2) + P(\bar{A}_2B_1) + P(\bar{A}_2B_2)\}] = -\Pi_B(1/2, 1/2).
\end{aligned} \tag{42}$$

We now insert from Eqs. (10) into the payoff (42) in order to express it in terms of the joint probability distribution P_{A_1,A_2,B_1,B_2} to obtain

$$\Pi_A(1/2,1/2) = P(A_1A_2B_1B_2) - P(A_1A_2\bar{B}_1\bar{B}_2) - P(\bar{A}_1\bar{A}_2B_1B_2) + P(\bar{A}_1\bar{A}_2\bar{B}_1\bar{B}_2). \tag{43}$$

As Bell's inequalities are assumed to hold, at this stage we refer to Fine's second theorem and insert the probability distribution P_{A_1,A_2,B_1,B_2} given by Eqs. (16,17) into Eq. (43) in order to re-express it in terms of joint probabilities:

$$\Pi_A(1/2,1/2) = [P(B_1) + P(B_2)][P(A_1) + P(A_2) - 1] - [(1+\beta)P(A_1) + (1-\alpha)P(A_2)]. \tag{44}$$

Now the requirement b) states that when Bell's inequalities hold, Alice's and Bob's payoffs for the strategy pair $(1/2,1/2)$ are both zero, giving us the second constraint on joint probabilities:

$$[P(B_1) + P(B_2)][P(A_1) + P(A_2) - 1] = [(1+\beta)P(A_1) + (1-\alpha)P(A_2)]. \tag{45}$$

After some manipulation, the constraints (38,39,45) can be re-expressed as

$$\begin{aligned}
P(A_1B_2) &= [(2+\beta)P(A_1) + P(B_1) + P(B_2) - \alpha P(A_2)]/2 - P(A_1B_1), && (46)\\
P(A_2B_1) &= [(1-\alpha)P(A_2) + 2P(B_1) + (1+\beta)P(A_1) - 2P(A_1B_1)]/2, && (47)\\
P(A_2B_2) &= [2P(A_1B_1) + P(A_2) + P(B_2) - P(A_1) - P(B_1)]/2. && (48)
\end{aligned}$$

To find if the violation of Bell's inequalities may lead to a NE, which is different from the classical NE of $(x^\star, y^\star) = (1/2,1/2)$, we consider Eqs. (31) to obtain

$$\begin{gathered}
\Pi_A(S_1,S_1') - \Pi_A(S_2,S_1') - \Pi_A(S_1,S_2') + \\
\Pi_A(S_2,S_2') = 4[P(A_1B_1) - P(A_2B_1) - P(A_1B_2)], \\
\Pi_A(S_1,S_2') - \Pi_A(S_2,S_2') = 2\{2[P(A_1B_2) - P(A_2B_2)] + \\
[P(A_2) - P(A_1)]\}.
\end{gathered} \tag{49}$$

We now substitute the constraints given by Eqs. (46,47,48) into (49) to obtain

$$\begin{gathered}
\Pi_A(S_1,S_1') - \Pi_A(S_2,S_1') - \Pi_A(S_1,S_2') + \\
\Pi_A(S_2,S_2') = 4[4P(A_1B_1) - (2+\beta)P(A_1) + \alpha P(A_2) - 2P(B_1)], \\
\Pi_A(S_1,S_2') - \Pi_A(S_2,S_2') = 2[(2+\beta)P(A_1) - 4P(A_1B_1) + 2P(B_1) - \alpha P(A_2)].
\end{gathered} \tag{50}$$

This allows to write the two inequalities in (30) as

$$\begin{aligned}
\Pi_A(x^\star,y^\star) - \Pi_A(x,y^\star) &= 4\Omega(y^\star - 1/2)(x^\star - x) \geq 0, && (51)\\
\Pi_B(x^\star,y^\star) - \Pi_B(x^\star,y) &= -4\Omega(x^\star - 1/2)(y^\star - y) \geq 0, && (52)
\end{aligned}$$

where

$$\Omega = 4P(A_1B_1) - (2+\beta)P(A_1) + \alpha P(A_2) - 2P(B_1). \tag{53}$$

Now, the inequalities (51,52) state that although the strategy pair $(x^\star, y^\star) = (1/2,1/2)$ remains a NE also in the quantum game, for which the system (12) of Bell's inequalities is violated, any pair of strategies will become a NE when $\Omega = 0$.

Cereceda reports in Ref. [53] that, corresponding to a maximally entangled bipartite state, there exist two sets of joint probabilities, which maximally violate the CHSH sum of correlations

while satisfying the constraints (6,7) that are given by normalization and causal communication constraint. To define these probability sets Cereceda divides the 16 joint probabilities into two sets:

$$\begin{aligned}\nu &= \left\{P(A_1\bar{B}_1), P(\bar{A}_1B_1), P(A_1\bar{B}_2), P(\bar{A}_1B_2), P(A_2\bar{B}_1), P(\bar{A}_2B_1), P(A_2B_2), P(\bar{A}_2\bar{B}_2)\right\}, \\ \mu &= \left\{P(A_1B_1), P(\bar{A}_1\bar{B}_1), P(A_1B_2), P(\bar{A}_1\bar{B}_2), P(A_2B_1), P(\bar{A}_2\bar{B}_1), P(A_2\bar{B}_2), P(\bar{A}_2B_2)\right\}.\end{aligned} \tag{54}$$

In terms of these the first probability set is then given as

$$P_l = (2+\sqrt{2})/8 \text{ for all } P_l \in \mu, \text{ and } P_m = (2-\sqrt{2})/8 \text{ for all } P_m \in \nu, \tag{55}$$

whereas the second probability set is given as

$$P_l = (2-\sqrt{2})/8 \text{ for all } P_l \in \mu, \text{ and } P_m = (2+\sqrt{2})/8 \text{ for all } P_m \in \nu. \tag{56}$$

These two sets, while being consistent with the normalization and causal communication constraints given by Eqs. (6,7), provide the maximum absolute limit of $2\sqrt{2}$ for the CHSH sum of correlations.

To evaluate Ω for these two probability sets we use the definition (13) to find γ for $n = 1, 2$ and $m \neq k = 1, 2$. We, therefore, consider the quantities

$$\begin{aligned}&[P(A_1B_1) + P(B_2) - P(A_1B_2), P(B_1), P(B_2)], \\ &[P(A_1B_2) + P(B_1) - P(A_1B_1), P(B_2), P(B_1)], \\ &[P(A_2B_1) + P(B_2) - P(A_2B_2), P(B_1), P(B_2)], \\ &[P(A_2B_2) + P(B_1) - P(A_2B_1), P(B_2), P(B_1)].\end{aligned} \tag{57}$$

The first of which, for example, is expressed as

$$\begin{aligned}&\{[P(A_1B_1) + P(A_1\bar{B}_1) + P(A_1B_2) + P(A_1\bar{B}_2)][P(A_1B_1) + P(\bar{A}_1B_1) + \\ &P(A_2B_1) + P(\bar{A}_2B_1)] + [P(A_1B_2) + P(\bar{A}_1B_2) + P(A_2B_2) + P(\bar{A}_2B_2)] - \\ &[P(A_1B_1) + P(A_1\bar{B}_1) + P(A_1B_2) + P(A_1\bar{B}_2)][P(A_1B_2) + P(\bar{A}_1B_2) + \\ &P(A_2B_2) + P(\bar{A}_2B_2)], [P(A_1B_1) + P(\bar{A}_1B_1) + P(A_2B_1) + P(\bar{A}_2B_1)], \\ &[P(A_1B_2) + P(\bar{A}_1B_2) + P(A_2B_2) + P(\bar{A}_2B_2)]\},\end{aligned} \tag{58}$$

which reduces itself to $\{1, 1, 1\}$ for the probability set (55). The same is the case with the remaining expressions of (57) for this probability set. So we obtain $\gamma = 1$ that gives $\Omega = 0$. Similarly, for the probability set (56) we also obtain $\Omega = 0$. In view of the Inequalities (51,52), in the quantum MP game with players sharing a maximally entangled state to play the game, this results in any strategy set $(x^\star, y^\star)$ being a NE. As for $\gamma = 1$ we obtain $\alpha = P(A_1)$ and $\beta = P(A_2)$ from Eq. (15), the constraints (46,47,48) are satisfied for both the probability sets in Eqs. (55,56).

7 Discussion

As the present paper builds up on our earlier work that constructs quantum games from non-factorizable joint probabilities [35], its brief review is in order. That work also uses the setting of a quantum correlation experiment and players's strategies are classical as is the case in the present approach. A classical game is re-expressed in terms of factorizable joint probabilities relevant to a shared physical system under the assumption that factorizable joint probabilities correspond to classicality. It is found that by requiring a classical outcome of the game to emerge

for factorizable joint probabilities results in constraints on the joint probabilities. These constraints ensure that the classical game corresponding to factorizable joint probabilities remains a subset of the quantum game. Retaining these constraints and allowing the joint probabilities to become non-factorizable leads to the corresponding quantum game. When played in this setting, it is found that new quantum mechanical NE emerge, for instance, for the game of Matching Pennies [44] that correspond, interestingly, to the sets of joint probabilities that maximally violate the CHSH inequality [52].

The relation, however, which this approach establishes between the classicality of the shared physical system, as expressed by a system of Bell's inequalities, and a classical game is not straightforward. It is because Bell's inequalities may not be violated even when the corresponding set of joint probabilities is non-factorizable i.e. non-factorizability is necessary but not sufficient to violate Bell's inequalities. A suggested explanation can be to state that such a game resides in the so-called pseudo-classical domain, where Bell's inequalities are not violated, and where a quantum game is treated as if players are simultaneously playing several classical games [27].

The present paper introduces a new approach to quantize a two-player two-strategy game. This approach, once again, uses the setting of quantum correlation experiments in which players' strategies remain classical. The quantum game is now obtained from the non-classical feature of the shared physical system consisting of the violation of Bell's inequalities. This situation corresponds when a joint probability distribution P_{A_1,A_2,B_1,B_2} does not exist, whose marginals are the joint probabilities P_{A_i,B_j}.

The argument presented in this paper can be described as follows. We begin by putting a classical game into a suitable format that permits us to consider the situation when a joint probability distribution P_{A_1,A_2,B_1,B_2}, whose marginals are the joint probabilities P_{A_i,B_j}, does not exist. We obtain this format by expressing players' payoff relations in terms of the joint probabilities P_{A_i,B_j}. With the payoff relations expressed in this way we select an arbitrary strategy pair $(x^\star, y^\star)$ and find constraints on the joint probabilities P_{A_i,B_j} that produce this NE and its corresponding classical payoffs to the players. Assuming that a joint probability distribution P_{A_1,A_2,B_1,B_2} exists that can be found from Fine's second theorem, we re-express the obtained constraints in terms of the joint probability distribution. We take the strategy pair $(x^\star, y^\star)$ to be the NE of the classical game, and players' payoffs for this strategy pair to be their payoffs in the classical game. This allows us to use Eqs. (16,17) and to express obtained constraints on the joint probability distribution P_{A_1,A_2,B_1,B_2} as constraints on joint probabilities P_{A_i,B_j}. The obtained constraints ensure that when the system of Bell's inequalities holds we obtain the classical NE as the outcome of the game. The quantum game is then obtained by retaining these constraints while allowing the joint probability distribution P_{A_1,A_2,B_1,B_2} not to exist. Considering a particular game we then investigate if this leads to NE that are non-classical.

We investigate games of Prisoners' Dilemma and Matching Pennies both of which have been studied earlier using other quantization schemes [5, 35, 44]. For PD we find that when the system of Bell's inequalities does not hold, and a joint probability distribution P_{A_1,A_2,B_1,B_2} does not exist, this cannot change or shift the classical NE of the game. For this game an identical result was reported in Ref. [35] using non-factorizable joint probabilities. For MP we find that in this quantization scheme the classical NE remains intact even when the system (12) of Bell's inequalities is violated and the joint probability distribution P_{A_1,A_2,B_1,B_2} does not exist. However, when a maximally entangled state is shared between players, any pair of strategies becomes a NE. This result diverges away from the one obtained earlier [44] using non-factorizable joint probabilities. We believe it is because non-factorizability is not equivalent to a joint probability distribution P_{A_1,A_2,B_1,B_2} not existing—a situation that motivates this paper. This analysis confirms that an outcome of a quantum game is dependent on the quantization route taken.

The results obtained show that for quantum games played in the setting of quantum correlation experiments the sharing of quantum resources does not always lead to players doing better than what they can do in the classical game. Players sharing a quantum system (for which Bell's inequalities are violated) do not automatically become better off relative to the ones who share classical system in order to physically implement the same game. This is observed to be the case

with the game of PD. On the contrary, for the game of MP the situation becomes quite different as players' sharing of the quantum system (that corresponds to a maximally entangled state) leads to the situation of any pair of strategies existing as NE. It, therefore, shows that the consequence of sharing of a quantum system depends on the particular original classical game the players play. This is in agreement with the results reported earlier by Shimamura et al. [23] showing that within Eisert et al.'s quantization scheme [5] certain quantum states cannot be used to quantize certain classical games as by doing so classical results cannot be reproduced. Our results convey the same message though the quantum games we consider are constructed directly from Bell's inequalities.

An interesting question is to ask about the consequence the new quantum solutions have regarding the original considered game. We believe that the new quantum solutions have a consequence regarding the original considered game only when a classical game emerges due to classicality of the shared physical system—in the sense that when the shared physical system does not violate Bell's inequalities the resulting game attains a classical interpretation (both in terms of the players' payoffs and the resulting pair of strategies defining the NE).

Some of the known criticisms of quantum games can be stated as follows: a) a quantum game is an ad-hoc construction that does not teach us anything new about quantum mechanics, b) it is known in game theory that if you change the rules of an old game, you get a new game. The fact that new NE appear in the quantum game is not surprising and does not imply anything about the old game, c) by including the new 'quantum' moves in a pay-off matrix, one can reformulate the quantum game as a purely classical game [10] with players having access to an extended set of available pure strategies.

In reply to a) we state that a quantum game offers a reasonable way to extend a classical game towards quantum domain and such an extension cannot, and should not, be assumed to open new avenues for quantum mechanics. However, it is the game theory for which a new avenue is opened in that taking a game to the quantum regime is found to have consequences for the outcome the considered game. In reply to b) we state that any construction of a quantum game is an extension of the original game and therefore the rules for playing this extended game also need to take into account the particular extension made. The rules of the extended game are relevant to the new game and therefore cannot be expected to remain identical to the ones in the original game. Under reasonable conditions, however, the extended game and its rules, should be reducible to the original game. In the construction we develop here, the reasonable constraints are a system of Bell's inequalities. In reply to c) we state that as in the present construction a quantum game corresponds only when Bell's inequalities, relevant to the shared physical system, are violated, and that the classical game corresponds when this system holds. As players' strategies remain classical in the quantum game, this offers the closest situation without changing the rules of the game—especially when compared to the usual case in which players' allowed strategy sets are extended to unitary transformations.

In the setting we use, the players strategies become dependent on the shared physical system via the payoff relations. For a shared system that exhibits quantum correlations, this may give the impression that the game is changed from a non-cooperative game to a cooperative game with different rules. We agree that it is possible to model our quantum game by introducing, for instance, pre-play negotiations into the standard setting of a non-cooperative game. Our objective, however, is different in that we ask how the violation of Bell's inequalities by the shared physical system impacts the outcome of a non-cooperative game. The possibility of modeling this impact by introducing some kind of cooperation between the players cannot be equated to changing a non-cooperative game into an explicitly cooperative game.

Agreeing with the earlier reported results [8, 9, 11, 24, 26, 29] this paper shows that players sharing a quantum mechanical system may or may result in new outcomes as this depends on the original classical game. Secondly, we present the first analysis that directly exploits a system of Bell's inequalities, together with Fine's results, establishing a direct link between a system of Bell's inequalities and the existence of a joint probability distribution in the construction of quantum games. Possible directions for further investigation may include expressing the outcome(s) of a quantum game in terms of the amount of the violation of Bell's inequalities such that they are

reduced to classical outcomes when there exists no violation. We believe the quantization approach proposed in this paper can be extended to multi-player games if Fine's results could be accordingly extended to more than 4 bivalent observables.

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References

[1] N.D. Mermin, Am. J. Phys. **58** (1990) 731.

[2] N.D. Mermin, Phys. Rev. Lett. **65** (1990) 1838.

[3] D.A. Meyer, Phys. Rev. Lett. **82** (1999) 1052.

[4] D.A. Meyer, quant-ph/0004092.

[5] J. Eisert, M. Wilkens, and M. Lewenstein, Phy. Rev. Lett. **83** (1999) 3077. See also, J. Eisert and M. Wilkens, J. Mod. Opt. **47** (2000) 2543.

[6] L. Vaidman, Found. Phys. **29** (1999) 615.

[7] H.E. Brandt, Prog. Quant. Elect. **22**/5 (1999) 257.

[8] S.C. Benjamin and P. M. Hayden, Phys. Rev. Lett. **87** (2001) 069801.

[9] S.C. Benjamin and P. M. Hayden, Phys. Rev. A **64** (2001) 030301.

[10] S.J. van Enk, R. Pike, Phys Rev A **66** (2002), 024306.

[11] N.F. Johnson, Phys. Rev. A **63** (2001) 020302.

[12] C.F. Lee and N.F. Johnson, Phys. Rev. A **67** (2003) 022311.

[13] L. Marinatto and T. Weber, Phys. Lett. A **272** (2000) 291.

[14] A. Iqbal and A.H. Toor, Phys. Lett. A **280** (2001) 249.

[15] J. Du, H. Li, X. Xu, X. Zhou, and R. Han, Phys. Lett. A **302** (2002) 229.

[16] A. Iqbal and A.H. Toor, Phys. Lett. A **286** (2001) 245.

[17] J. Du, H. Li, X. Xu, M. Shi, J. Wu, X. Zhou and R. Han, Phys. Rev. Lett. **88** (2002) 137902.

[18] E.W. Piotrowski and J. Sladkowski, Physica A **312** (2002) 208.

[19] A. Iqbal and A.H. Toor, Phys. Lett. A **294** (2002) 261.

[20] E.W. Piotrowski and J. Sladkowski, Int. J. Theor. Phys. **42** (2003) 1089.

[21] A.P. Flitney and D. Abbott, Fluct. Noise Lett. **2** (2002) R175.

[22] A. Iqbal and A.H. Toor, Phys. Rev. A **65** (2002) 022306.

[23] J. Shimamura, S. K. Özdemir, F. Morikoshi, and N. Imoto, Phys. Lett. A **328** (2004) 20.

[24] S.K. Özdemir, J. Shimamura, F. Morikoshi, N. Imoto, Phys. Lett. A, **333** (2004) 218.

[25] A.P. Flitney and D. Abbott, J. Phys. A **38** (2005) 449.

[26] R. Mendes, Quantum Information Processing, **4** (2005) 1.

[27] T. Cheon and I. Tsutsui, Phys. Lett. A **348** (2006) 147.

[28] A. Nawaz and A.H. Toor, J. Phys. A, Math. Gen. **37** (2004) 11457.

[29] J. Shimamura, S.K. Özdemir, F. Morikoshi and N. Imoto, Int. J. Quant. Inf. **2** (2004) 79.

[30] A. Iqbal and A.H. Toor, Comm. Theo. Phys. **42** (2004) 335.

[31] T. Ichikawa and I. Tsutsui, Annals of Physics **322** (2007) 531.

[32] T. Cheon and A. Iqbal, J. Phys. Soc. Japan, **77** (2008) 024801.

[33] A. Iqbal and S. Weigert, J. Phys. A **37** (2004) 5873.

[34] S.K. Özdemir, J. Shimamura, and N. Imoto, New J. Phys. **9** (2007) 43.

[35] A. Iqbal and T. Cheon, Phys. Rev. E **76** (2007) 061122.

[36] M. Ramzan and M.K. Khan, J. Phys. A, Math. Theor. **41** (2008) 435302.

[37] A.P. Flitney and L.C.L. Hollenberg, Phys. Lett. A **363** (2007) 381.

[38] S.K. Özdemir, J. Shimamura, N. Imoto, New J. Phys. **9** (2007) 43

[39] A. Iqbal and T. Cheon, Proc. SPIE. **6802** (2007) 68020A.

[40] S.A. Bleiler, arXiv:0808.1389 and arXiv: 0808.1391

[41] A. Iqbal and T. Cheon, Evolutionary stability in quantum games, Chapter 13 in *Quantum Aspects of Life*, edited by D. Abbott, P.C.W. Davies and A.K. Pati, Imperial College Press (2008).

[42] T. Ichikawa, I. Tsutsui, and T. Cheon, J. Phys. A, Math. Theor. **41** (2008) 135303.

[43] A. Iqbal, T. Cheon, and D. Abbott, Phys. Lett. A **372** (2008) 6564.

[44] A. Iqbal and D. Abbott, J. Phy. Soc. Japan **78**/1 (2009) 014803.

[45] C. Schmid, A.P. Flitney, W. Wieczorek, N. Kiesel, H. Weinfurter, and L.C.L. Hollenberg, arXiv:0901.0063

[46] F.S. Khan, Quantum Multiplexers, Parrondo Games, and Proper Quantization, Doctoral dissertation, Portland State University (2009) arXiv:0906.0645

[47] S.A. Bleiler, arXiv:0902.2196

[48] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47** (1935) 777.

[49] D. Bohm, *Quantum Theory*, (Prentice-Hall, Englewood Cliffs, NJ, 1951), pp. 611–623.

[50] J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).

[51] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49** (1982) 1804.

[52] A. Peres, *Quantum Theory*: *Concepts and Methods* (Kluwer Academic Publishers, 1995).

[53] J.L. Cereceda, Found. Phys. Lett. **13** (2000) 427.

[54] K. Binmore, *Game Theory: A Very Short Introduction* (Oxford University Press, USA, 2007).

[55] E. Rasmusen, *Games & Information*: *An Introduction to Game Theory* (Blackwell Publishers Ltd., Oxford, 2001) 3rd Edition.

[56] D. Abbott, P.C.W. Davies and C.R. Shalizi, Fluct. Noise Lett. **2**/4 (2002), C1.

[57] J. Maynard Smith and G. R. Price, Nature **246** (1973) 15.

[58] J.W. Weibull, *Evolutionary Game Theory*, The MIT Press, Cambridge (1995).

[59] L. Samuelson, *Evolutionary Games and Equilibrium Selection*, the MIT Press (1997).

[60] N. Gisin, Phys. Lett. A **154** (1991) 201.

[61] P. Busch, P. Kienzler, P. Lahti, and P. Mittelstaedt, Phys. Rev. A **47** (1993) 4627.

[62] B.S. Cirel'son, Lett. Math. Phys. **4** (1980) 93.

[63] A. Fine, J. Math. Phys. **23** (1982), 1306.